MATH 105A and 110A Review: Gram-Schmidt process

Facts to Know:

Let $\mathcal{B}$ be the collection of k vectors in $\mathbb{R}^n$:

$$\mathcal{B} = \{ V_1, \dots, V_k \}.$$

$\mathcal{B}$ is said to be **orthogonal** if

$$V_i \cdot V_j = 0 \quad \text{for every } i \neq j.$$

$\mathcal{B}$ is said to be **orthonormal** if

$$V_i \cdot V_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Given a basis $\mathcal{B}$ for a subspace S of $\mathbb{R}^n$, then we can use the **The Gram-Schmidt process** to find another basis for S that is orthonormal.

The **projection operator** is defined by

$$\operatorname{proj}_u x = \frac{u \cdot x}{u \cdot u} u$$

Gram-Schmidt process for two vectors: Let v_1, v_2 be a basis for a some subspace of $\mathbb{R}^n$.

1. Set $u_1 = V_1$ Then set $w_1 = \frac{1}{\sqrt{u_1 \cdot u_1}} u_1$
2. Set $u_2 = V_2 - \operatorname{proj}_{u_1} V_2$ Then set $w_2 = \frac{1}{\sqrt{u_2 \cdot u_2}} u_2$

Examples:

1. The basis

$$\mathcal{B} = \left\{ \underset{V_1}{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}}, \underset{V_2}{\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}} \right\}$$

is a basis for a plane in $\mathbb{R}^3$. Find an orthonormal basis the same plane.

$$u_1 = V_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \Rightarrow w_1 = \frac{1}{\sqrt{1+4+4}} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$u_2 = v_2 - \text{proj}_{u_1} v_2$$

$$= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \frac{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}}{1+4+4} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \cancel{\frac{-1+4}{9}}\ \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3/3 - 1/3 \\ 0 - 2/3 \\ 6/3 - 2/3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} \\ -2/3 \\ 4/3 \end{pmatrix}$$

$$w_2 = \frac{1}{\sqrt{\frac{16}{9} + \frac{4}{9} + \frac{16}{9}}} \begin{pmatrix} -4/3 \\ -2/3 \\ 4/3 \end{pmatrix}$$

$$= \frac{1}{\frac{2}{\cancel{3}}\sqrt{4+1+4}} \begin{pmatrix} -4/3 \\ -2/3 \\ 4/3 \end{pmatrix}$$

$$= \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}.$$

The orthogonal basis is

$$\{w_1, w_2\},$$

where

$$w_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$w_2 = \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$

Checking our work

$$w_1 \cdot w_2 = \frac{-2}{9} - \frac{2}{9} + \frac{4}{9} = 0$$

$$w_1 \cdot w_1 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1$$

$$w_2 \cdot w_2 = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1.$$